Dilations of partial representations

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1. Dilations of partial representations on sets, modules and algebras

2. Dilations of partial representations on Hilbert spaces

3. Dilations of certain interaction groups

1. Representations of $k_{par}(G) \longleftrightarrow$ partial representations of G.

2. Representations of a partial crossed product $\longleftrightarrow$ covariant representations of the corresponding partial system.

3. Partial representations $\neq$ partial actions.

4. Dilation of partial representations should be related with globalization of partial actions.

5. Application of the existence of enveloping actions: if $v : G \to B(H)$ is a partial representation of a discrete group G on a Hilbert space H, there exist:
   - a Hilbert space K which contains H as a Hilbert subspace,
   - a unitary representation $u : G \to B(K)$,
   - an orthogonal projection $P : K \to H$,

   such that $v_t = Pu_t|_H$, $\forall t \in G$.

6. Interaction groups.

Dilation of partial representations on sets, modules and algebras

1. A partial action α on a commutative C^*-algebra has an enveloping action if and only if the graph of $\hat{\alpha}$ is closed (FA, 2003).
2. Corollary: a partial action on a commutative and unital C^*-algebra has an enveloping action iff every ideal of the partial action is unital.
3. A partial action on a unital algebra has an enveloping action iff every ideal of the partial action is unital (Dokuchaev-Exel, 2005).
4. Observation: *in the proof of the latter result the authors dilate the partial representation* $v : G \to \mathrm{End}_{alg}(A)$ *such that* $v_t(a) = \alpha_t(a1_{t^{-1}})$.

Definition

A partial homomorphism of a group G into a monoid S is a map $v : G \to S$ such that

- $v_e = 1_S$, *where e is the unit element of G.*
- $v_r v_{r^{-1}} v_s = v_r v_{r^{-1}s}, \forall r, s \in G$
- $v_r v_{s^{-1}} v_s = v_{rs^{-1}} v_s, \forall r, s \in G$

When $S = \mathrm{End}_{\mathscr{C}}(X)$, for certain object X in a category $\mathscr{C}$, we say that v is a partial representation of G on the object X.

Proposition (Structure of a partial representation)

Let $v : G \to \mathrm{End}_{Sets}(X)$ be a partial representation, $X_r := v_r(X)$, $\pi_r := v_r v_{r^{-1}}$, and $\alpha_r : X_{r^{-1}} \to X_r$ such that $\alpha_r(x) = v_r(x)$, $\forall r \in G$. Then:

1. $\pi_r^2 = \pi_r$, $\pi_r(X) = X_r$, *and* $\pi_r \pi_s = \pi_s \pi_r$, $\forall r, s \in G$.
2. $v_r = \alpha_r \pi_{r^{-1}}$, $\forall r \in G$.
3. $\alpha := (\{X_r\}, \{\alpha_r\})$ *is a partial action of G on X.*

Dilations of partial representations

Example (model)

Let Y *be a set,* $u : G \to \mathrm{Aut}_{Sets}(Y)$ *a representation of* G *on* Y*, and* $Q : Y \to Y$ *such that* $Q^2 = Q$*. Suppose in additon that* $Q_r Q_s = Q_s Q_r$*,* $\forall r, s \in G$*, where* $Q_r = u_r Q u_{r^{-1}}$*. If* $X := Q(Y)$ *and* $v : G \to \mathrm{End}_{Sets}(X)$ *is given by* $v_t := Qu_t|_X$*. Then* v *is a partial representation of* G *on* X*.*

Definition

Let $\mathscr{C}$ *be a category. We define a category* $\mathbb{T}_{\mathscr{C}}$*.*

Objects: (Y, u, Q) *such that:* $Y \in Ob(\mathscr{C})$*,* $u : G \to \mathrm{Aut}_{\mathscr{C}}(Y)$*,* $Q \in \mathrm{End}_{\mathscr{C}}(Y)$ *is such that* $Q^2 = Q$ *and* $Q_r Q_s = Q_s Q_r$*,* $\forall r, s \in G$*.*

Morphisms: $\varphi : (Y, u, Q) \to (Y', u', Q')$ *is* $\varphi \in \mathrm{Hom}_{\mathscr{C}}(Y, Y')$ *such that* $\varphi u_t = u'_t \varphi$ *and* $\varphi Q = Q' \varphi$*.*

Definition

Let $\mathscr{C}$ be one of the following categories: Sets, R-mod, or k-algebras. A dilation of a partial representation $v : G \to \mathrm{End}_{\mathscr{C}}(X)$ is a pair (j, T) such that:

1. $T = (Y, u, Q) \in Ob(\mathbb{T}_{\mathscr{C}})$
2. $j \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ *is injective, and* $Q(Y) = j(X)$. *In the case of k-algebras we also require that* $j(X)$ *is an ideal of* Y.
3. $jv_t = Qu_tj, \forall t \in G$.

The dilation is called minimal when Y *is generated by* $\bigcup_{t \in G} u_t(j(X))$.

Remark

In k-algebras, the condition $j(X) \lhd Y$ *implies* $v_t(X) \lhd X, \forall t$.

Proposition

If $(j, (Y, u, Q))$ *is a dilation of a partial representation* $v = \pi\alpha$, *then* $\alpha \overset{j}{\cong} u|_{j(X)}$. *If the dilation is minimal,* u *is a minimal globalization of* α.

Proof.

If $x \in X_{t^{-1}}$, then

$$j\alpha_t(x) = Qu_t j(x) = QQ_t u_t j(x) = Q_t Qu_t j(x) \in u_t(j(X)) \cap j(X).$$

Now if $j(y) = u_t j(z) \in j(X) \cap u_t j(X)$:

$$y = v_t(z) \qquad \text{and} \qquad j(z) = Qj(z) = Qu_{t^{-1}} j(y) = jv_{t^{-1}}(y)$$

It follows $y = \alpha_t(z)$, $j(X_t) = u_t(j(X)) \cap j(X)$ and $j(\alpha_t(z)) = u_t j(z)$. □

Theorem

Any partial representation $v : G \to \mathrm{End}_{Sets}(X)$ admits a dilation. If the dilation $D = (j, T)$ is minimal, then for any other dilation $D' = (j', T')$ there exists a unique morphism $\varphi : D \to D'$. In particular any two minimal dilations of v are isomorphic.

Proof.

- $\bar{X} = \{y : G \to X\}$.
- $\rho : G \times \bar{X} \to \bar{X}$ such that $\rho_t(y)(s) := y(st)$.
- $j : X \to \bar{X}$: $j(x)(t) := v_t(x)$, and $Y = \bigcup_{t \in G} \rho_t(j(X))$.
- $Q : Y \to Y$: $Q(y) := j(y(e))$, and $u_t = \rho_t|_Y$.
- $j(X_t) = j(X) \cap u_t(j(X))$
- $u_t j(x) = j\alpha_t(x)$, $\forall x \in X_{t^{-1}}$ and $t \in G$.
- $QQ_t(y)|_s = v_s v_t(y(t^{-1}))$ and $Q_t Q(y)|_s = v_{st} v_{t^{-1}}(y(e))$.
- $QQ_t(u_r j(x))|_s = v_s v_t v_{t^{-1}r}(x)$ and $Q_t Q(u_r j(x))|_s = v_{st} v_{t^{-1}} v_r(x)$.

Then:

- $QQ_t = Q_t Q$ on Y
- $Qu_t j = j(u_t(j(x))(e)) = j(j(x)(t)) = j(v_t(x)) = jv_t(x)$

The existence and uniqueness of the map φ follows from the universal property of the enveloping action. Finally:

$$Q'\varphi u_t j = Q' u'_t \varphi j = Q' u'_t j' = j' v_t = \varphi j v_t = \varphi Q u_t j$$

Dilations of partial representations on modules

Theorem

Any partial representation $v : G \to \mathrm{End}_R(M)$ *admits a minimal dilation* $(j,(N,u,Q))$ *which is* **faithful**, *i.e.:* $Qu_t(n) = 0\ \forall t \in G$ *implies* $n = 0$*. In this case* $u|_{j(M)} \cong \alpha$*, where* α *is the partial action associated with* v*.*
Moreover, if D *is a faithful and minimal dilation of* v *and* $D' = (j',(N',u',Q'))$ *is another dilation of* v*, there exists a unique morphism* $\varphi : D' \to D$ *such that* $\varphi j' = j$*.*
In particular any two minimal and faithful dilations of v *are isomorphic.*

Proof.

- $\bar{M} = \{y : G \to M\}$ with its natural structure of R-module.
- $\rho : G \times \bar{M} \to \bar{M}$ such that $\rho_t(y)(s) := y(st)$.
- $j : M \to \bar{M}$: $\quad j(m)(t) := v_t(m)$, and $N = \text{span}_{t \in G}\, \rho_t(j(M))$.
- $Q : N \to N$: $\quad Q(m) := j(m(e))$, and $u_t = \rho_t|_N$.

Then:

- $Qu_t j = jv_t$, $\forall t \in G$, and $\alpha \cong u|_{j(M)}$; therefore (j, u, N) is a minimal globalization of α in the category of R-modules.
- $(j, (N, u, Q))$ is faithful because $Q\rho_t(y) = j(y(t))$.

If $D' = (j', (N', u', Q'))$ is another dilation of v, and $\sum_{t \in G} u'_t j'(m_t) = 0$:

$$0 = Q' u'_r (\sum_{t \in G} u'_t j'(m_t)) = \sum_{t \in G} j'(v_{rt}(m_t)) = j'(\sum_{t \in G} v_{rt}(m_t)), \forall r \in G.$$

Then: $0 = j(\sum_{t \in G} v_{rt}(m_t)) = Qu_r(\sum_{t \in G} u_t(j(m_t)))$, $\forall r$.

Define $\varphi : N' \to N$ such that $\varphi(\sum_t u'_t j'(m'_t)) := \sum_t u_t j(m'_t)$. □

Dilations of partial representations on algebras

Theorem

Let $v : G \to \mathrm{End}_{k-alg}(A)$ *be a partial representation such that* $v_t(A) \lhd A$, $\forall t \in G$. *Then* v *admits a minimal dilation* (j, T) *which is faithful (again:* $Qu_t(b) = 0\ \forall t \in G$ *implies* $b = 0$*) and such that* $u|_{j(A)} \cong \alpha$, *where* α *is the partial action associated with* v.
Moreover, if $D = (j, T)$ *is a faithful and minimal dilation of* v *and* $D' = (j', T')$ *is another dilation of* v, *there exists a unique morphism* $\varphi : T' \to T$ *such that* $\varphi j' = j$.
In particular any two minimal and faithful dilations of v *are isomorphic.*

Proof.

- Consider again: $\bar{A} = \{y : G \to A\}$ with its natural structure of k-algebra; $\rho : G \times \bar{A} \to \bar{A}$ such that $\rho_t(y)(s) := y(st)$; $j : A \to \bar{A}$: $j(a)(t) := v_t(a)$.
- Define $B := \langle \rho_t(j(A)) : t \in G \rangle$, $Q : B \to B$: $Q(b) := j(b(e))$, and $u_t = \rho_t|_B$.
- We have $j(A) \lhd B$ and $B = \operatorname{span}_{t \in G} \rho_t j(A)$:

$$u_t(a)j(a')|_s = v_s v_{s^{-1}}(v_{st}(a)v_s(a')) = v_s(v_t(a)a') = j(v_t(a)a')|_s$$

$$j(a)u_t j(a')|_s = v_s v_{s^{-1}}(v_s(a)v_{st}(a')) = v_s(av_t(a')) = j(av_t(a'))|_s$$

As in the case of partial representations on modules, $(j, (B, u, Q))$ is a faithful and minimal dilation of v, $\alpha \cong u|_{j(A)}$; (j, u, B) is a minimal globalization of α in the category of k-algebras, and if $D' = (j', (B', u', Q'))$ is another dilation of v, the map $\varphi : T' \to T$ such that $\varphi(\sum_t u'_t j'(a_t)) := \sum_t u_t j(a_t)$ is a homomorphism that satisfies $\varphi j' = j$. □

Dilations of partial representations on Hilbert spaces

Recall:

1. $C_p^*(G) = C(X) \rtimes_\alpha G$, where
   - $X_t = \{x : G \to \{0,1\} / \, x(e) = 1 = x(t)\}$, $X = X_e$.
   - $\alpha_t : X_{t^{-1}} \to X_t$: $\alpha_t(x)|_s = x(t^{-1}s)$
2. (α, X) has enveloping action (β, Y): $Y = \{y : G \to \{0,1\} : \; y \neq 0\}$ -which is a Hausdorff space- and β given by the same formula as α.
3. If $E^*(G) := C(Y) \rtimes_\beta G$, then $E^*(G) \overset{MR}{\sim} C_p^*(G)$ (Morita-Rieffel equivalence). In fact $C_p^*(G)$ is a hereditary subalgebra of $E^*(G)$.

Then:

partial rep. v of G $\longmapsto$ (*corresponds to*) unital rep. ρ of $C_p^*(G)$

partial rep. v of G ⋮ (*dilates*) unitary rep. u of G

unital rep. ρ of $C_p^*(G)$ $\downarrow$ (*induces*) unital rep. $\tilde{\rho}$ of $E^*(G)$

unital rep. $\tilde{\rho}$ of $E^*(G)$ (*decomposes as*) $\longmapsto$ cov. rep. $(\tilde{\pi}, u)$ $\longmapsto$ unitary rep. u of G

The Morita-Rieffel equivalence $C^*_p(G) \overset{MR}{\sim} E^*(G)$ follows from:

Theorem (FA 2003: reduced case; FA & Laura Martí 2009: full case)

Let $=(B_t)_{t\in G}$ *be a Fell bundle,* $\mathcal{E} = (E_t)_{t\in G}$ *a right ideal of* $\mathcal{B}$*, and* $\mathcal{A} = (A_t)_{t\in G}$ *a sub-Fell bundle of* $\mathcal{B}$ *contained in* $\mathcal{E}$ *such that*

1. $\mathcal{A}\mathcal{E} = \mathcal{E}$.
2. $\mathcal{E}\mathcal{E}^* \subseteq \mathcal{A}$.

Then $C^*_{red}(\mathcal{A})$ *is a hereditary subalgebra of* $C^*_{red}(\mathcal{B})$ *and* $C^*(\mathcal{A})$ *is a hereditary subalgebra of* $C^*(\mathcal{B})$*. If, moreover,* $\overline{\mathrm{span}}(B_t \cap \mathcal{E}^*\mathcal{E}) = B_t$*,* $\forall t \in G$*, then* $C^*_{red}(\mathcal{A}) \overset{MR}{\sim} C^*_{red}(\mathcal{B})$ *and* $C^*(\mathcal{A}) \overset{MR}{\sim} C^*(\mathcal{B})$*.*

Corollary (FA 2003: reduced case; FA & L. Martí 2009: full case)

If $\beta G \times B \to B$ *is a globalization of the partial action* α *on* A*, then* $A \rtimes_{red,\alpha} G$ *is a hereditary subalgebra of* $B \rtimes_{red,\beta} G$*, and* $A \rtimes_\alpha G$ *is a hereditary subalgebra of* $B \rtimes_\beta G$*. If* β *is the enveloping action of* α*, then* $A \rtimes_{red,\alpha} G \overset{MR}{\sim} B \rtimes_{red,\beta} G$ *and* $A \rtimes_\alpha G \overset{MR}{\sim} B \rtimes_\beta G$*.*

Dilations of certain interaction groups

Definition

An interaction group is a triple (A, G, V) *where* A *is a unital* C^**-algebra,* G *is a group, and* V *is a map from* G *into* $\mathrm{B}(A)$*, which satisfies:*

1. V_t *is a positive unital map,* $\forall t \in G$.
2. V *is a partial representation.*
3. $V_t(ab) = V_t(a)V_t(b)$ *if either* a *or* b *belongs to* $V_{t^{-1}}(A)$.

Example

Let X *be a compact Hausdorff space and* $\theta : X \to X$ *a surjective continuous map. Consider the unital injective endomorphism* $\alpha : C(X) \to C(X)$ *induced by* θ*:* $\alpha(a) = a \circ \theta$*. Suppose there exists a unital transfer operator for* α*, i.e., a positive linear map* $\mathcal{L} : C(X) \to C(X)$ *such that* $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, $\forall a, b \in C(X)$*. Then* $V : \mathbb{Z} \to B(C(X))$ *such that* $V_n = \begin{cases} \alpha^n & n \geq 0 \\ \mathcal{L}^{-n} & n \leq 0 \end{cases}$ *is an interaction group.*

Example

Suppose $F : B \to B$ *is a conditional expectation with range* A, $\beta : G \times B \to B$ *is an action such that* $F_r F_s = F_s F_r$, $\forall r, s \in G$, *where* $F_r = \beta_r F \beta_{r^{-1}}$. *If* $F\beta_r F(1) = 1$, $\forall r$, *then* $V : G \to B(A)$ *such that* $V_t = F\beta_t|_A$ *is an interaction group.*

Theorem

Let P *be a submonoid of a group* G *such that* $G = P^{-1}P$, *and let* α *be an action of* P *by unital injective endomorphisms of the unital* C^**-algebra* A, *and suppose* $V : G \to \mathrm{B}(A)$ *is an interaction group such that* $V|_P = \alpha$. *Then* V *has a minimal dilation* (i, T), *where* $T = (B, \beta, F)$ *and* $i : A \to B$ *is an embedding, which has the following universal property. If* $(i', (B', \beta', F'))$ *is another dilation of* V, *then there exists a unique morphism* $\phi : (B, \beta, F) \to (B', \beta', F')$ *such that* $\phi i = i'$. *Therefore the dilation* (i, T) *is unique up to isomorphism.*

Proof

Theorem (Marcelo Laca, 2000)

There exists a C^*-dynamical system (B, G, β), unique up to isomorphism, consisting of an action β of G by automorphisms of a C^*-algebra B and an embedding $i : A \to B$ such that:

1. *β dilates α, that is, $\beta_t \circ i = i \circ \alpha_t$, for t in P, and*
2. *(B, G, β) is minimal, that is, $\bigcup_{t \in P} \beta_t^{-1}(i(A))$ is dense in B.*

There is a partial order in P: $r \leq s \iff s = tr$, for some $t \in P$. Suppose $r, s \in P$, with $r \leq s$, and $a_r, a_s \in A$ are such that $\beta_{r^{-1}}(a_r) = \beta_{s^{-1}}(a_s)$, then $\beta_{sr^{-1}}(a_r) = a_s$, so $\alpha_{sr^{-1}}(a_r) = a_s$. Then:

$$V_{s^{-1}}(a_s) = V_{s^{-1}}\alpha_{sr^{-1}}(a_r) = V_{s^{-1}}V_{sr^{-1}}(a_r) = V_{s^{-1}}\alpha_s V_{r^{-1}}(a_r) = V_{r^{-1}}(a_r).$$

Define $F : B \to B$ such that $F(b) = V_{t^{-1}}(\beta_t(b))$, $\forall b \in \beta_{t^{-1}}(A)$. Suppose $t = r^{-1}s \in G$, with $r, s \in P$. We have

$$F\beta_t|_A = F\beta_{r^{-1}}\beta_{rt}|_A = F\beta_{r^{-1}}\alpha_s = V_{r^{-1}}\alpha_s = V_{r^{-1}}V_r V_{r^{-1}s} = V_{r^{-1}s} = V_t$$

Example (Exel-Renault interaction groups; 2007)

Suppose there is a cocycle for the action θ, *that is, a continuous map* $\omega : P \times X \to (0,1]$ *that satisfies*

1. $\sum_{y \in \theta_t^{-1}(x)} \omega(t,y) = 1.$
2. $\omega(rs,x) = \omega(r,x)\omega(s,\theta_r(x))$
3. $\omega(s,x)W_r(C_{x,y}^{s,r}) = \omega(r,x)W_s(C_{x,y}^{r,s})$

Then there is an interaction group $V^\omega : G \to B(C(X))$ *such that, if* $t = r^{-1}s$, $r,s \in P$, *then* $V_t^\omega(a) = \sum_{y \in \theta_r^{-1}(x)} \omega(r,y)a(\theta_s(y))$. *The cocycle can be interpreted as an inverse system of measures, whose limit is a measure that defines the conditional expectation* F.

Example (Iterated function systems, G. de Castro, 2009)

$\gamma, \gamma_1, \ldots, \gamma_d : X \to X$ *continuous, such that* $\gamma\gamma_i = id_X$, $\forall i$. *If* α *and* α_i *are the endomorphisms induced by* γ *and* γ_i *on* $A := C(X)$, *then* $\mathcal{L} := \frac{1}{d}\sum_{i=1}^{d} \alpha_i$ *is a transfer operator for* α. *Then we have an interaction group* $V : \mathbb{Z} \to B(A)$.

Example (IFS+Exel-Renault interaction group)

When $X = \biguplus_{i=1}^{d} \gamma_i(X)$ *("strong separation condition"),* V *is an Exel-Renault interaction group, with cocycle* $\omega(n, y) = 1/d^n$. *We may suppose* $X = \{1, \ldots, d\}^{\mathbb{N}}$, $\gamma(x)(j) = x(j+1)$,

$$\gamma_i(x)(j) = \begin{cases} i & \text{if } j = 0 \\ x(j-1) & \text{if } j \geq 1 \end{cases}.$$

Let $Y := \{1, \ldots, d\}^{\mathbb{Z}}$, $\tilde{\gamma} : Y \to Y$ *such that* $\tilde{\gamma}(y)(j) = y(j+1)$, *and* $\pi : Y \to X$ *the restriction,* $B = C(Y)$, $\beta : B \to B$ *the dual map of* $\tilde{\gamma}$, *and* $i : A \to B$ *the dual map of* π. *Note that* $\pi\tilde{\gamma}^n = \gamma^n\pi$, $\forall n \in \mathbb{N}$. *Next define*

$$\tau_i : X \to Y \text{ such that } \tau_i(x)(j) = \begin{cases} i & \text{if } j < 0 \\ x(j) & \text{if } j \geq 0 \end{cases}.$$

Then $\pi\tau_j = id_X$ *and* $\rho_j i = id_A$, *where* ρ_j *is the dual map of* τ_j. *Define* $F_j, F : B \to B$ *as* $F_j := i\rho_j$, *and* $F = \frac{1}{d}\sum_{j=1}^{d} F_j$. *Then* F *is a conditional expectation with range* $i(A)$, *and* $F\beta^n i = iV_n$, $\forall n \in \mathbb{Z}$.

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